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Entry flow in a channel

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A uniformly valid asymptotic solution for large Reynolds number is constructed for plane steady laminar flow of a liquid into the channel between two semi-infinite parallel plates. The entry condition is taken as either that for a cascade of plates in a uniform oncoming stream, or uniform flow directly at the inlet. A paradox in the standard solution of Schlichting—that near the inlet the flow due to displacement would not be the accelerated uniform core on which his expansion is based—is resolved by showing that his series for small as well as large distance actually applies only to conditions far downstream, and matches with another expansion valid near the inlet. Good agreement is found with three independent numerical solutions of the full Navier–Stokes equations, except for a discrepancy in one solution for uniform entry that is traced to erroneous neglect of inlet vorticity.

1. Introduction

The development of a parabolic Poiseuille profile downstream of entry into a plane channel is one of the standard problems in laminar-flow theory. It has attracted more attention than is warranted by its intrinsic practical importance, because it exemplifies certain general features of viscous flow. It therefore appears in textbooks, and is continually re-examined as new phenomena are introduced. Thus it has been extended to axisymmetric flow (Atkinson & Goldstein 1938), tested for stability (Hahneman, Freeman & Finston 1948), modified for magnetohydrodynamics (Shercliff 1956), for a non-Newtonian liquid (Collins & Schowalter 1963), for suction or injection through porous walls (Horton & Yuan 1964), for a viscoelastic fluid (Metzner & White 1965), for a compressible fluid (Blankenship & Chung 1967), and for a tube of general cross-section (McComas 1967).

Most approximate analyses of the problem (approaching 100 in number) involve some form of Prandtl's boundary-layer approximation. Four general methods of solution may be discerned in the literature: (1) numerical finite-difference solution of the boundary-layer equations (initiated by Bodoia & Osterle 1961), (2) linearization of the inertia terms (Boussinesq 1891; Langhaar 1942), (3) integral methods (Schiller 1922), and (4) series expansions (Schlichting 1934).

We are concerned here with understanding an assumption common to the last two methods: that the flow consists of boundary layers near the walls together with a central inviscid core in which the velocity increases downstream to satisfy continuity but is uniform and parallel at each station (figure 1). This last feature seems to present a paradox, which is particularly evident in the analysis of Schlichting. He solves the boundary-layer equations by expanding the stream function in a series for small distances from the inlet, starting with the Blasius solution, and matching with the pressure distribution for a uniform accelerating core. However, the displacement effect of the boundary layers induces a change in the inviscid flow that would obviously, near the inlet, have no resemblance to a uniform core. It would approach a uniform core far downstream, but for large distances Schlichting (following Boussinesq) constructs another asymptotic expansion by perturbing the final Poiseuille flow and patching with the first series at an intermediate station.

The details of the inviscid core near the entrance will depend upon the entry conditions. Two models have been used in the literature: (1) uniform parallel flow at entry, which is supposed to approximate the situation following a rapid well-rounded contraction, and (2) uniform flow far upstream, with the channel walls extended upstream as streamlines, which corresponds to an infinite cascade of equally-spaced plates in a uniform oncoming stream (figure 1).

We seek a uniformly valid approximation for large Reynolds number under both entry conditions. In so doing, we resolve the paradox of the uniform core by showing that Schlichting's entire analysis represents only a 'downstream' analysis, which matches (in the sense of the method of matched asymptotic expansions) with an 'upstream' expansion valid near the inlet.†

Happily, finite-difference solutions of the full Navier–Stokes equations have been carried out for these two entry conditions at one Reynolds number by Wang & Longwell (1964), and for uniform entry at several Reynolds numbers by Gillis & Brandt (1964) (see also Brandt & Gillis 1966). The agreement with our solution is good except in Wang & Longwell's case of uniform entry, and that discrepancy has been traced to the fact that they overlooked vorticity at the inlet.

2. Cascade in uniform oncoming stream

Schlichting's classical paper is a model of brevity; and we seek comparable conciseness by explaining our solution largely in physical terms, leaving the interested reader to restore the straightforward mathematical details. We start with the case of uniform flow far upstream of a cascade of plates, which is the simpler entry condition.

Near the inlet it is natural to refer lengths to the half-width a of the channel, and velocities to the free-stream speed U (figure 1). We call these *upstream variables*, and use them to construct an *upstream expansion* valid near the inlet. [We disregard a tiny circular neighbourhood of each leading edge—region O of figure 1—where the characteristic reference length is ν/U. There a first approximation is the solution of the full Navier–Stokes equations for a semi-infinite plate, which has been calculated approximately by Davis (1967).] We define the Reynolds number as $R = Ua/\nu$.

† The upstream expansion has been studied much more thoroughly by Wilson (1970). His independent analysis came to our attention while the present paper was in proof.

As R becomes infinite (but the motion remains steady and laminar!) the flow tends to the uniform stream for all bounded x (region I of figure 1) except at the surface of each plate (region II). There it approaches the Blasius flat-plate boundary layer, so that near the upper and lower plates of figure 1 the (dimensionless) stream function is

$$\psi \sim \pm[1-R^{-\frac{1}{2}}(2x)^{\frac{1}{2}}f(\eta)], \quad \eta = R^{\frac{1}{2}}(1\mp y)/(2x)^{\frac{1}{2}}. \tag{2.1}$$

Here f is the Blasius function in the Falkner–Skan normalization (Rosenhead 1963, p. 222).

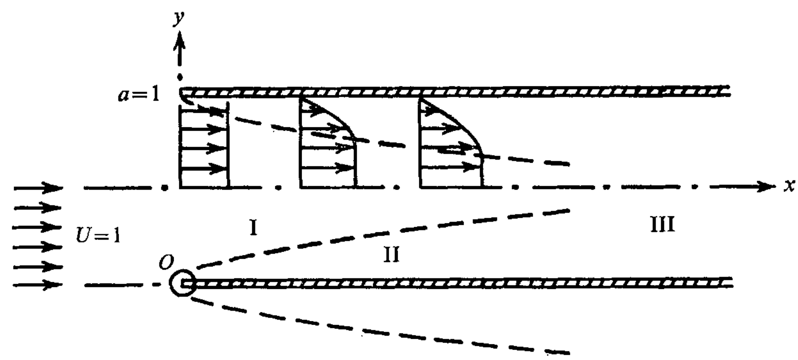

FIGURE 1. Notation in upstream variables. The conventional uniform-core model is shown above centreline, and conditions for cascade of plates in uniform oncoming stream below. The upstream region consists of the inviscid core I and boundary layers II; the downstream region is marked III.

2.1. *Flow due to displacement*

A second approximation in the inviscid core (region I) is found by substituting $\psi = y+R^{-\frac{1}{2}}\psi_2(x,y)+\ldots$ into the Navier–Stokes equations. Here ψ_2 represents the flow due to displacement. In this problem it is a potential flow that vanishes far upstream, and has on each plate surface the outward normal velocity $\beta(2x)^{-\frac{1}{2}}$ induced by the Blasius boundary layer, where $\beta = 1{\cdot}21678$.

We have solved this potential problem by distributing sources along each plate of the cascade, and equivalently by applying the Fourier transformation from the viewpoint of generalized functions (Lighthill 1958). The source method is the simpler for calculating velocities at finite x. It gives the streamwise velocity increment in region I as

$$u_2 = \psi_{2y} = 2^{-\frac{3}{2}}\beta\sum_{n=1}^{\infty}\left[2^{\frac{3}{2}}(n^{\frac{1}{2}}-(n-1)^{\frac{1}{2}})-\left\{\frac{[x^2+(2n-1+y)^2]^{\frac{1}{2}}-x}{x^2+(2n-1+y)^2}\right\}^{\frac{1}{2}} - \left\{\frac{[x^2+(2n-1-y)^2]^{\frac{1}{2}}-x}{x^2+(2n-1-y)^2}\right\}^{\frac{1}{2}}\right]. \tag{2.2}$$

Here the second and third terms are the contributions of the nth plate below and above the centreline $y = 0$, and the first term serves to restore the velocity to zero far upstream. The sum has been evaluated by computer using 50 and 100 terms.

The velocity profile across the inlet is shown in figure 2 for $R = 75$. The upstream expansion, consisting of the uniform stream plus the flow due to displacement, agrees surprisingly well with the full numerical solution, and differs significantly from a uniform core. This inviscid approximation applies across the entire inlet if the boundary layer is calculated in Cartesian co-ordinates. However, the approximation is further improved near the plate by forming a multiplicative composite expansion (Van Dyke 1964) as the inviscid expansion times the Blasius boundary-layer solution in parabolic co-ordinates divided by their common element, which is $[1-\frac{1}{2}\beta R^{-\frac{1}{2}}(1\pm y)^{-\frac{1}{2}}]$. Parabolic co-ordinates have been used to allow upstream influence of viscosity; they are optimal for a single plate (Kaplun 1954) and therefore presumably nearly optimal for the cascade.

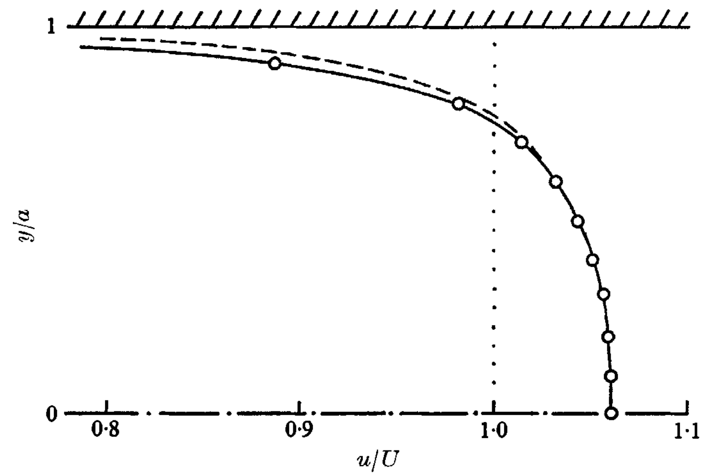

FIGURE 2. Streamwise velocity profile across inlet to cascade of plates at $R = 75$ . . ., uniform core (Schlichting 1934); ———, 2-term upstream expansion; ———, composite upstream expansion using boundary layer in parabolic co-ordinates; ○, full numerical solution (Wang & Longwell 1964).

Although the inviscid velocity profile is convex at the inlet, the source solution (2.2) shows that it becomes concave on the centreline at about $x = 0{\cdot}35$. Profiles downstream of that station will consequently have maximum velocity off the centreline until the peaks are eroded by the thickening boundary layers. This effect was first noticed by Wang & Longwell (1964) and Gillis & Brandt (1964) in their numerical solutions.

The velocity along the centreline is shown in figure 3. Near the inlet, the upstream expansion agrees well with the full numerical solution. From comparison also with the downstream approximation discussed below, it appears to remain valid several half-widths downstream.

The approximation obviously deteriorates downstream. Its asymptotic behaviour can be found by recalculating the flow due to displacement using the Fourier transformation with respect to x, which gives for the transform of the stream function

$$\bar{\Psi}_2(y;s) = \frac{\beta}{8\pi}\frac{i\,\mathrm{sgn}\,s-1}{|s|^{\frac{3}{2}}}\frac{\sinh(2\pi sy)}{\sinh(2\pi s)}. \tag{2.3}$$

Expanding for small s and inverting term-by-term using table I of Lighthill (1958) yields, for $x\rightarrow\pm\infty$,

$$u_2 \sim \beta\left[|x|^{\frac{1}{2}}+\tfrac{1}{24}(3y^2-1)\frac{1}{|x|^{\frac{3}{2}}}+\ldots\right]\frac{1+\mathrm{sgn}\,x}{2}. \tag{2.4}$$

This expression evidently ceases to be valid at distances so great that x is of order R (in dimensional terms, at distances of order Ua^2/ν). A physical interpretation is that the displacement thickness of the Blasius boundary layers would fill the channel at $x = \beta^{-2}R$.

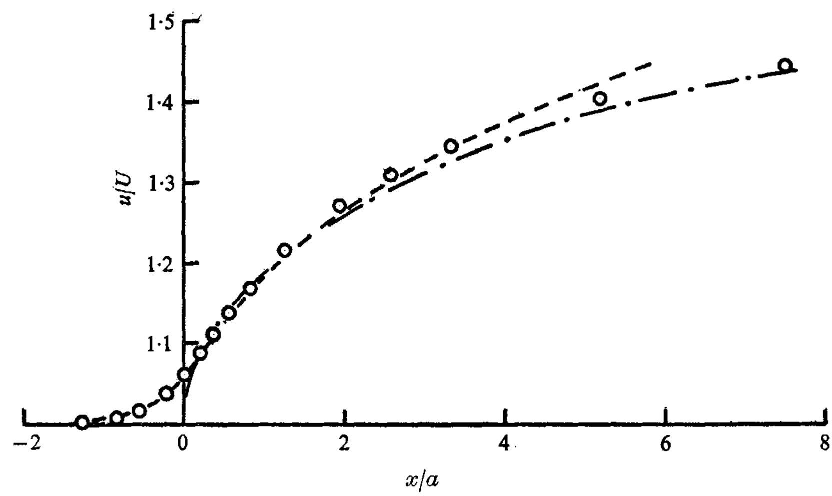

FIGURE 3. Velocity along centreline for cascade of plates at $R = 75$. ———, 2-term upstream expansion; —·—·, 1-term downstream expansion (Bodoia & Osterle 1961); ○, full numerical solution (Wang & Longwell 1964).

2.2. *Downstream approximation*

The non-uniformity for large x suggests introducing a contracted *downstream variable* $\xi = x/R$ (which is $\nu x/Ua^2$ in dimensional terms), and thereby seeking a *downstream expansion* to complement the upstream expansion in its region of invalidity—region III of figure 1. Our variable ξ is the square of Schlichting's expansion quantity ϵ. In downstream variables the Navier–Stokes equations give for the stream function

$$\left(\psi_y \frac{\partial}{\partial \xi} - \psi_\xi \frac{\partial}{\partial y}\right)(\psi_{yy} + R^{-2}\psi_{\xi\xi}) = \psi_{yyyy} + 2R^{-2}\psi_{\xi\xi yy} + R^{-4}\psi_{\xi\xi\xi\xi}. \tag{2.5}$$

This suggests that the downstream expansion will proceed in powers of R^{-2}.

Setting $R = \infty$ shows that the first approximation is a solution of Prandtl's boundary-layer equations, with the streamwise pressure distribution unknown. [In contrast to classical boundary-layer theory, the transverse dimension is here $O(1)$ rather than $O(R^{-\frac{1}{2}})$ and the transverse velocity is $O(R^{-1})$ rather than $O(R^{-\frac{1}{2}})$. The classical orders reappear, however, in Schlichting's expansion for small ξ.] Three ξ derivatives have been lost, so the double upstream and downstream boundary conditions of the full equation reduce to the single condition of matching with the upstream expansion. (The solution will automatically approach the Poiseuille flow downstream.) The asymptotic matching principle (Van Dyke 1964) asserts that the one-term upstream expansion of the one-term downstream expansion should equal the one-term downstream expansion of the one-term upstream expansion, which is simply y. Thus the problem for the first downstream approximation is

$$\frac{\partial}{\partial y}(\psi_{1yyy}-\psi_{1y}\psi_{1\xi y}+\psi_{1\xi}\psi_{1yy}) = 0, \tag{2.6a}$$

$$\psi_{1\xi}(\xi,\pm 1) = \psi_{1y}(\xi,\pm 1) = 0, \tag{2.6b}$$

$$\psi_1(0,y) = y. \tag{2.6c}$$

We recognize this as the problem undertaken by Schlichting. It was later solved numerically by Bodoia & Osterle (1961), and we adopt their results as being more accurate. Figure 3 shows the velocity on the centreline for $R = 75$, taken from the tables of Bodoia (1959). The transition from the upstream to the downstream approximation is so smooth that it is scarcely necessary to combine them into a uniformly valid composite expansion. Agreement with the full numerical solution is reasonably good throughout.

Schlichting (1934) attacked the problem by expanding for small and large ξ. [Additional terms were calculated by Collins & Schowalter (1962) and Roidt & Cess (1962).] The expansion for small ξ is once more a singular perturbation, involving thin boundary layers on the plates, and the accelerating but uniform core that is now seen to be acceptable because it actually applies only far from the inlet. Thus we have resolved the paradoxical aspect of Schlichting's solution by identifying it as the expansion, for small contracted abscissa, of an approximation that is valid only far downstream.

The second term in the downstream expansion would match with the second term in (2.4). This confirms that the correction would be of relative order R^{-2}. Thus our one-term downstream approximation is more accurate than our two-term upstream expansion, which has an error of relative order R^{-1} even in the inviscid core. The factor $(3y^2-1)$ in (2.4) means that the inviscid core would no longer be uniform far downstream in the second approximation.

3. Uniform entry

We turn now to the case, more frequently discussed in the literature, of uniform parallel flow at the inlet. The general structure of the asymptotic solution is the same; and we dwell only on some novel features that arise because weak vorticity is generated at the inlet.

3.1. *Flow due to displacement*

The flow outside a boundary layer is usually irrotational, as in the previous case, but in general it is only inviscid. Thus again substituting $\psi = y+R^{-\frac{1}{2}}\psi_2(x,y)+\ldots$ into the Navier–Stokes equations shows that the flow due to displacement is governed by the linearized Euler equations:

$$\frac{\partial}{\partial x}\nabla^2\psi_2 = 0 \quad \text{or} \quad \nabla^2\psi_2 = -\omega(y). \tag{3.1}$$

(To this order the vorticity is constant along the streamlines $y =$ const. of the basic uniform flow.) Whereas vorticity was absent in the previous case, it must now be present in order to satisfy the conditions $u = 1$ and $v = 0$ at the inlet. That is, some mechanism such as a mesh of varying porosity would have to be imposed across the inlet to maintain uniform parallel inflow against the small pressure gradients induced by boundary-layer displacement. [Such a mesh has actually been used to approximate a flat initial profile in the axisymmetric experiments of Atkinson, Zdzislaw & Smith (1967) and others.]

The general solution of (3.1) is

$$\psi_2 = \psi_{2p}(x, y) + F(y), \tag{3.2}$$

where ψ_{2p} is a potential function, and the arbitrary function F contains all the vorticity. We have solved the potential problem by adding to the previous system of source distributions a system of images of reversed sign reflected in the inlet plane $x = 0$ in order to maintain $v = 0$ there. This gives a streamwise velocity increment

$$\begin{aligned} u_{2p} = 2^{-\frac{3}{2}}\beta \sum_{n=1}^{\infty} \Bigg[& 2^{\frac{5}{2}}(n^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}) - \left\{\frac{[x^2+(2n-1+y)^2]^{\frac{1}{2}} - x}{x^2+(2n-1+y)^2}\right\}^{\frac{1}{2}} \\ & - \left\{\frac{[x^2+(2n-1-y)^2]^{\frac{1}{2}} - x}{x^2+(2n-1-y)^2}\right\}^{\frac{1}{2}} - \left\{\frac{[x^2+(2n-1+y)^2]^{\frac{1}{2}} + x}{x^2+(2n-1+y)^2}\right\}^{\frac{1}{2}} \\ & - \left\{\frac{[x^2+(2n-1-y)^2]^{\frac{1}{2}} + x}{x^2+(2n-1-y)^2}\right\}^{\frac{1}{2}} \Bigg]. \end{aligned} \tag{3.3}$$

We then take the additional component $F'(y)$ as the negative of the value of this velocity at the inlet, in order to make the total increment vanish there. Thus the streamwise velocity due to displacement is given by

$$u_2 = u_{2p}(x, y) - u_{2p}(0, y). \tag{3.4}$$

The resulting velocity distribution along the centreline for $R = 75$ is shown in figure 4. It disagrees with the numerical solution of Wang & Longwell (1964) even close to the inlet. Andreas Acrivos suggested that the discrepancy might result from neglect of inlet vorticity in the numerical solution. Although no evidence appears in the published work, it was found that in his thesis Wang (1963) does indeed replace the original inlet conditions $u = 1$, $v = 0$ by the erroneous set $\psi = y$, $\omega = 0$. (It is possible, however, that the latter conditions would actually correspond more closely to entry from a well-rounded contraction.) Gillis & Brandt (1964) [cf. Brandt & Gillis 1966] have avoided this error, and their velocity distributions have the inflected shape of our result.

Solving the displacement problem instead by Fourier transformation shows that far downstream the velocity is now given asymptotically by

$$u \sim 1 + R^{-\frac{1}{2}}[\beta x^{\frac{1}{2}} - u_{2p}(0, y) + \tfrac{1}{24}\beta(3y^2 - 1)x^{-\frac{3}{2}} + \ldots]. \tag{3.5}$$

Thus the upstream expansion is again invalid for $x = O(R)$.

3.2. *Downstream expansion*

The leading term of the downstream expansion is the same as in the previous case, because the matching condition (2.6*c*) arises from the term proportional to $x^{\frac{1}{2}}$ that appears in (3.5) as well as (2.4). The centreline velocity from Gillis &

Brandt's (1964) full numerical solution† is seen in figure 4 to depart from our upstream expansion after a few half-widths, and to tend toward that downstream approximation.

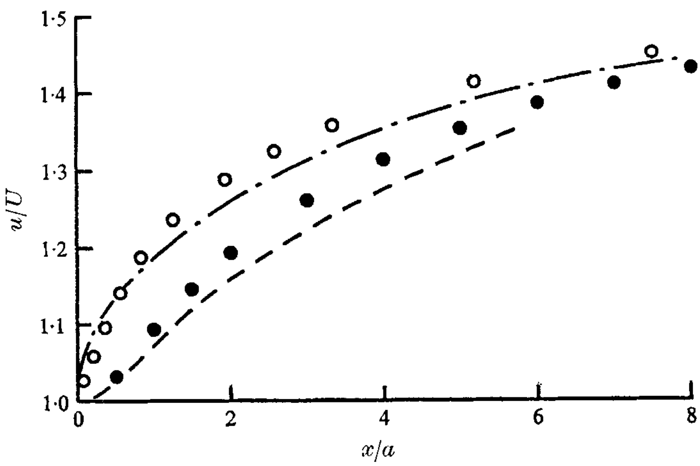

FIGURE 4. Velocity along centreline for uniform entry at $R = 75$. ———, 2 term upstream expansion; —·—·, 1 term downstream expansion (Bodoia & Osterle 1961); ○, full numerical solution (Wang & Longwell 1964); ●, full numerical solution (Gillis & Brandt, interpolated from $R = 25, 50, 100$).

The accuracy of the downstream approximation is less in this case, because the secondary term is no longer as small as order R^{-2}. Rather, a term of order $R^{-\frac{1}{2}}$ is required in order to match the new rotational component $u_{2p}(0, y)$ in (3.5), and non-linearity then requires an expansion in successive powers of $R^{-\frac{1}{2}}$. Setting $\psi = \psi_1(\xi, y) + R^{-\frac{1}{2}}\psi_2(\xi, y) + \ldots$ in the downstream equation (2.5) shows that the problem for the secondary term is

$$\frac{\partial}{\partial y}(\psi_{2yyy} - \psi_{1y}\psi_{2\xi y} - \psi_{2y}\psi_{1\xi y} + \psi_{1\xi}\psi_{2yy} + \psi_{2\xi}\psi_{1yy}) = 0, \tag{3.6a}$$

$$\psi_2(\xi, \pm 1) = \psi_{2y}(\xi, \pm 1) = 0, \tag{3.6b}$$

$$\psi_2(0, y) = u_{2p}(0, y). \tag{3.6c}$$

Just as the leading downstream term is a solution of Prandtl's boundary-layer equations, so the secondary term is a solution of the conventional second-order boundary-layer equations (Van Dyke 1969), with the pressure distribution unknown in each case. The solution could be calculated numerically by perturbing the solution of Bodoia & Osterle (1961), or by expanding in series following Schlichting (1934).

4. Discussion

Uniform entry produces an interesting—though perhaps unrealistic—complication that is absent from the cascade. The inviscid shear flow in the upstream expansion and the second term ψ_2 in the downstream expansion are two different

† Values for $R = 75$ have been obtained by interpolation in the solutions for $R = 25$, 50, and 100.

views of a weak wake that forms behind whatever mechanism is imagined to enforce uniform parallel flow at the inlet. Because the Reynolds number is high, viscous diffusion of the wake is negligible in the upstream section, and asserts itself only over the long distance to the downstream region.

Whereas the inlet condition on the transverse velocity component is adjusted by a potential flow within a distance of the order of the channel width, the longitudinal velocity is equilibrated only much more slowly in the wake. This is in accord with the general principle in boundary-layer theory that the thinner layer takes care of the higher normal derivative. Our wake is the counterpart in viscous flow of the 'wide layer' encountered by Johnson & Reissner (1960) in the linear theory of elasticity for the bending of a semi-infinite plate. It is similar also to the 'wake behind a two-dimensional grid' discovered by Kovasznay (1948), which can represent large as well as small disturbances because it is simultaneously a solution of the Navier–Stokes, Oseen, and boundary-layer equations. Our problem is more involved than these, however, in that the central flow interacts with the boundary layers on the walls.

A further complication appears in higher approximations for uniform entry. In the upstream region the inviscid shear velocity $-u_{2p}(0, y)$ is seen from (3.3) to be singular at the upper and lower plates like

$$-2^{-\frac{1}{2}}\beta R^{-\frac{1}{2}}(1 \mp y)^{-\frac{1}{2}}.$$

Thus in the second approximation the boundary layer lies beneath an inviscid flow that is infinite at the surface. Boundary layers under singular external conditions have recently been examined by Conti & Van Dyke (1969), who show that the regular progression in powers of $R^{-\frac{1}{2}}$ is interrupted by the intrusion of other powers (and logarithms) of Reynolds number. Here the consequence is that the second term of the boundary-layer expansion in the upstream region would differ from the first by only $R^{-\frac{1}{4}}$.†

The scale of the entry layer is R times the channel width in any case. One may ask what is the scale of the corresponding 'exit layer': if a fully-developed Poiseuille flow is modified at some station, how far do the disturbances spread upstream? Whereas Prandtl boundary layers are $O(R^{-\frac{1}{2}})$ thick, Oseen boundary layers (except on a surface parallel to the basic stream) are only $O(R^{-1})$ thick. This suggests that if the exit disturbance is small the viscous effect will extend upstream only a distance R^{-1} times the channel width; and this is the case even for large disturbances in the second solution of Kovasznay (1948). However, the direct viscous effect will ordinarily be accompanied by an inviscid rotational disturbance, so that the actual disturbance will extend upstream a few channel widths. This conclusion is confirmed by Wilson's (1969) recent calculations of eigensolutions for small upstream perturbations of plane Poiseuille flow.

It might be useful to extend the present analysis to axisymmetric flow. All the required elements are available: Atkinson & Goldstein (1938) have treated

† Wilson (1970) has shown that the further course of the expansion is even more surprising: the third- and each higher-order term differs from its predecessor by order $R^{-\frac{1}{8}}$, $R^{-\frac{1}{16}}$, $R^{-\frac{1}{32}}$, etc., so that an infinite number terms—with a 'point of condensation'—separate the first-order boundary-layer solution from the usual correction of relative order $R^{-\frac{1}{2}}$.

the downstream approximation by series expansions, and Hornbeck (1964) has calculated it numerically. The full Navier–Stokes equations have been solved numerically by Friedmann, Gillis & Liron (1968) for uniform entry, and by Vrentas, Duda & Bargeron (1966) for uniform flow far upstream with the pipe walls extended upstream as free streamlines (a rather more artificial situation than in plane motion.) Likewise, the other generalizations of channel entry mentioned in the introduction would gain in clarity if not in accuracy under reconsideration from the present viewpoint.

I am indebted to Keith Stewartson for a lively discussion that clarified the singular-perturbation nature of this problem, to Andreas Acrivos for a brilliant piece of detective work in deducing from scanty clues that vorticity must have been overlooked in the numerical solution for uniform entry at $R = 75$, and to David Kassoy and Joseph Keller for helpful comments. Professors F. Osterle and C. S. Pings kindly supplied theses that were otherwise unavailable. This work was supported by Air Force Office of Scientific Research Contract F 44620-69-C-0036.

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